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Covariant representations of Hilbert C^*-modules

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Abstract

We show that a full Hilbert C^*-module X can be embedded in the multiplier module of the crossed product of X by a continuous action η of a locally compact group G on X. Also, we show that there is a bijective correspondence between nondegenerate covariant representations of X and nondegenerate representations of the crossed product of X by η, and moreover, this correspondence preserves the irreducibility and unitary equivalence.

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1. Introduction

Hilbert C^*-modules are generalizations of Hilbert spaces and C^*-algebras, such many concepts from the theory of Hilbert spaces and the theory of C^*-algebras were extended to the context of Hilbert C^*-modules, for example, the notion of representation of a C^*-algebra on a Hilbert space [1] or the notion of continuous action of a locally compact group on a C^*-algebra [2–5]. Hilbert C^*-modules are also useful tools in the theory of operator algebras, operator K-theory, KK-theory of C^*-algebras, group representation theory, C^*-algebraic theory of quantum groups and theory of operator spaces. For example, Hilbert

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C^*-modules appear as imprimitivity bimodules in the study of the Morita equivalence for C^*-algebras. Given two C^*-dynamical systems (G, α, A) and (G, β, B) such that A and B are strongly Morita equivalent and an (α, β)-compatible action η of G on the imprimitivity $A - B$ bimodule X, then the C^*-crossed products $G \times_\alpha A$ and $G \times_\beta B$ are strongly Morita equivalent [2,3], the imprimitivity $G \times_\alpha A - G \times_\beta B$ bimodule is called the crossed product of X by η and it is denoted by $G \times_\eta X$. In [5], Kusuda proved a duality theorem for crossed products of Hilbert C^*-modules by actions of locally compact groups and a duality theorem for crossed products of Hilbert C^*-modules by coactions of locally compact groups.

In this paper, we show that given a dynamical system on Hilbert C^*-modules, (G, η, X), X can be embedding in the multiplier module of $G \times_\eta X$ and G is isomorphic to a group of unitaries in the C^*-algebra of adjointable module morphisms on $G \times_\eta X$. Also, we introduce the notion of covariant representation of a dynamical system on Hilbert C^*-modules and prove that there is a bijective correspondence between nondegenerate covariant representations of (G, η, X) and nondegenerate representations of $G \times_\eta X$ that preserves the irreducibility and unitary equivalence.

2. Preliminaries

A *Hilbert C^*-module* X over a C^*-algebra A (or a Hilbert A-module) is a linear space that is also a right A-module, equipped with an A-valued inner product $\langle \cdot, \cdot \rangle$ that is $\mathbb{C}$- and A-linear in the second variable and conjugate linear in the first variable such that X is complete with the norm $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$. X is *full* if the closed bilateral $*$-sided ideal $\langle X, X\rangle$ of A generated by $\{\langle x, y\rangle; x, y \in X\}$ coincides with A.

Any C^*-algebra A is a Hilbert A-module with A-valued inner product given by $\langle a, b\rangle = a^*b$.

Given two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, the Banach space $L(\mathcal{H}, \mathcal{K})$ of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$, has a natural structure of Hilbert C^*-module over the C^*-algebra $L(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$ with the action on $L(\mathcal{H})$ on $L(\mathcal{H}, \mathcal{K})$ given by $T \cdot S = TS$ for $T \in L(\mathcal{H}, \mathcal{K})$ and $S \in L(\mathcal{H})$, and $L(\mathcal{H})$-valued inner product given by $\langle T_1, T_2\rangle = T_1^* T_2$.

A morphism of Hilbert C^*-modules is a map $\Psi : X \to Y$ from a Hilbert A-module X to a Hilbert B-module Y with the property that there is a C^*-morphism $\psi : A \to B$ such that

$$\langle \Psi(x), \Psi(y)\rangle = \psi(\langle x, y\rangle)$$

for all x and y in X. A map $\Psi : X \to Y$ is *an isomorphism of Hilbert C^*-modules* if it is invertible, and if Ψ and Ψ^{-1} are morphisms of Hilbert C^*-modules.

A representation of X on the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ is a morphism of Hilbert C^*-modules π_X from X to the Hilbert $L(\mathcal{H})$-module $L(\mathcal{H}, \mathcal{K})$. If X is full, then the $*$-representation π_A associated to π_X is unique. A representation $\pi_X : X \to L(\mathcal{H}, \mathcal{K})$ of X is *nondegenerate* if $[\pi_X(X)\mathcal{H}] = \mathcal{K}$ and $\left[\pi_X(X)^*\mathcal{K}\right] = \mathcal{H}$ (here, $[Y]$ denotes the closed subspace of a Hilbert space Z generated by the subset $Y \subseteq Z$). Two representations $(\pi_X, \mathcal{H}, \mathcal{K})$ and $\left(\pi'_X, \mathcal{H}', \mathcal{K}'\right)$ are *unitarily equivalent* if there are two unitary operators $U_1 \in L(\mathcal{H}, \mathcal{H}')$ and $U_2 \in L(\mathcal{K}, \mathcal{K}')$ such that $U_2\pi_X(x) = \pi'_X(x)U_1$ for all x in X. A pair $(\mathcal{H}_0, \mathcal{K}_0)$ of Hilbert subspaces, $\mathcal{H}_0 \subseteq \mathcal{H}$ and $\mathcal{K}_0 \subseteq \mathcal{K}$, is invariant under $(\pi_X, \mathcal{H}, \mathcal{K})$

if $\pi_X(X)\mathcal{H}_0 \subseteq \mathcal{K}_0$ and $\pi_X(X)^*\mathcal{K}_0 \subseteq \mathcal{H}_0$. A representation $(\pi_X, H, \mathcal{K})$ is *irreducible* if $(\{0\}, \{0\})$ and $(\mathcal{H}, \mathcal{K})$ are the only pairs invariant under $(\pi_X, \mathcal{H}, \mathcal{K})$.

The C^*-algebra $L(A)$ of all adjointable module morphisms on A can be identified to the multiplier algebra $M(A)$ of A (see, for example, [6]), and the vector space of all adjointable module morphisms from A to X has a natural structure of Hilbert $M(A)$-module (see, for example, [7]). It is denoted by $M(X)$ and is called the multiplier module of X [7]. The *strict topology* on $M(X)$ is given by the family of seminorms $\{\|\cdot\|_{a,x}\}_{(a,x)\in A\times X}$, where $\|h\|_{a,x} = \|h(a)\| + \|h^*(x)\|$.

Lemma 2.1. *Let X be a full Hilbert C^*-module and let π_X be a nondegenerate representation of X on the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. Then π_X extends to a nondegenerate representation $\overline{\pi_X}$ of $M(X)$ on the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. Moreover, if π_X is irreducible, then $\overline{\pi_X}$ is irreducible, and if $(\pi_X, \mathcal{H}, \mathcal{K})$ and $\left(\pi'_X, \mathcal{H}', \mathcal{K}'\right)$ are unitarily equivalent, then $(\overline{\pi_X}, \mathcal{H}, \mathcal{K})$ and $\left(\overline{\pi'_X}, \mathcal{H}', \mathcal{K}'\right)$ are unitarily equivalent.*

Proof. Let $\{x_i\}_{i\in I}$ be a bounded net which converges strictly to 0. Then $\{\langle x_i, x_i\rangle\}_{i\in I}$ converges strictly to 0, since $\|\langle x_i, x_i\rangle\, a\| = \|\langle x_i, x_i a\rangle\| \le \|x_i\|\,\|x_i a\|$ and $\{x_i\}_{i\in I}$ is bounded. Then, since π_A is nondegenerate [1, Lemma 3.4], the net $\{\pi_A(\langle x_i, x_i\rangle)\}_{i\in I}$ converges strictly to 0, and from

$$\|\pi_X(x_i)h\|^2 = \|\langle h, \pi_A(\langle x_i, x_i\rangle)\, h\rangle\| \le \|h\|\,\|\pi_A(\langle x_i, x_i\rangle)\, h\|$$

for all $h \in \mathcal{H}$, we deduce that the net $\{\pi_X(x_i)h\}_{i\in I}$ converges to 0 for all $h \in \mathcal{H}$.

On the other hand, the net $\{\pi_X(x_i)^*\pi_X(y)h\}_{i\in I}$ converges to 0 for all $h \in \mathcal{H}$ and for all $y \in X$, since

$$\pi_X(x_i)^*\pi_X(y)h = \pi_A(\langle x_i, y\rangle)h$$

and the net $\{\langle x_i, y\rangle\}_{i\in I}$ converges to 0 for all $y \in X$. From this fact, and taking into account that π_X is nondegenerate, we deduce that the net $\{\pi_X(x_i)^*k\}_{i\in I}$ converges to 0 for all $k \in \mathcal{K}$. Such we showed that the net $\{\pi_X(x_i)\}_{i\in I}$ converges strictly to 0.

Let $z \in M(X)$. Then the net $\{ze_i\}_{i\in I}$ from X, where $\{e_i\}_{i\in I}$ is an approximate unit for A, is bounded and it converges strictly to z. We define $\overline{\pi_X}(z)h = \lim_i \pi_X(ze_i)h$ for all $h \in \mathcal{H}$. Since

$$\begin{aligned}
\langle \overline{\pi_X}(z_1), \overline{\pi_X}(z_2)\rangle\,(h) &= \lim_j \left(\lim_i \left\langle \pi_X(z_1 e_j), \pi_X(z_2 e_i)\right\rangle (h)\right)\\
&= \lim_j \left(\lim_i \pi_A\left(e_j\,\langle z_1, z_2\rangle\, e_i\right)(h)\right)\\
&= \lim_j \pi_A\left(e_j\,\langle z_1, z_2\rangle\right)(h) = \overline{\pi_A}\left(\langle z_1, z_2\rangle\right)(h)
\end{aligned}$$

where $\overline{\pi_A}$ is the extension of the $*$-representation π_A to $M(A)$, for all $h \in \mathcal{H}$ and for all $z_1, z_2 \in M(X)$, $\overline{\pi_X}$ is a representation of $M(X)$ on the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. Moreover, $\overline{\pi_X}$ is nondegenerate. Clearly, if π_X is irreducible, then $\overline{\pi_X}$ is irreducible, and if $(\pi_X, \mathcal{H}, \mathcal{K})$ and $\left(\pi'_X, \mathcal{H}', \mathcal{K}'\right)$ are unitarily equivalent, then $(\overline{\pi_X}, \mathcal{H}, \mathcal{K})$ and $\left(\overline{\pi'_X}, \mathcal{H}', \mathcal{K}'\right)$ are unitarily equivalent. $\square$

Suppose that G is a locally compact group, Δ is the modular function of G with respect to the left invariant Haar measure ds. *A continuous action* of G on a full Hilbert A-module X is a group morphism $t \mapsto \eta_t$ from G to $\mathrm{Aut}(X)$, the group of all isomorphisms of Hilbert C^*-modules from X to X, such that the map $t \mapsto \eta_t(x)$ from G to X is continuous for each $x \in X$. The triple (G, η, X) will be called a *dynamical system on Hilbert C^*-modules*. Clearly, any C^*-dynamical system (G, α, A) can be regarded as a dynamical system on Hilbert C^*-modules in the sense of the above definition.

Any continuous action $t \mapsto \eta_t$ of G on X induces a unique continuous action $t \mapsto \alpha_t^\eta$ of G on A such that $\alpha_t^\eta(\langle x, y\rangle) = \langle \eta_t(x), \eta_t(x)\rangle$ for all $x, y \in X$ and for all $t \in G$ [4].

3. Covariant representations

Let (G, η, X) be a dynamical system on Hilbert C^*-modules and let Y be a Hilbert C^*-module over a C^*-algebra B.

Definition 3.1. A covariant morphism from X to $M(Y)$ is a triple (v, Φ, u) consisting of a morphism of Hilbert C^*-modules Φ from X to $M(Y)$, a strict continuous group morphism u from G to $\mathcal{U}(M(B))$ and a strict continuous group morphism v from G to $\mathcal{U}(M(K(Y)))$, where $K(Y)$ is the C^*-algebra of all compact operators on Y, such that

$$v_t \Phi(x) u_{t^{-1}} = \Phi(\eta_t(x))$$

for all $x \in X$ and for all $t \in G$. If $[\Phi(X)B] = Y$ and $\left[\Phi(X)^*Y\right] = B$, we say that the covariant morphism (v, Φ, u) is nondegenerate.

Remark 3.2. The notion of covariant morphism of Hilbert C^*-modules extends the notion of covariant morphism of C^*-algebras. Indeed, if (G, α, A) is a C^*-dynamical system and (φ, u) is a (nondegenerate) covariant morphism from A to $M(B)$, then (u, φ, u) is a (nondegenerate) covariant morphism from the Hilbert C^*-module A to the Hilbert C^*-module $M(B)$.

Remark 3.3. If (v, Φ, u) is a (nondegenerate) covariant morphism from X to $M(Y)$, then (φ, u) is a (nondegenerate) covariant morphism from A to $M(B)$.

Indeed, we have

$$\begin{aligned}\varphi\left(\alpha_t^\eta \langle x, y\rangle\right) &= \varphi(\langle \eta_t(x), \eta_t(y)\rangle) = \langle \Phi(\eta_t(x)), \Phi(\eta_t(y))\rangle \\ &= u_t(\Phi(x))^* \Phi(y) u_{t^{-1}} = u_t \varphi(\langle x, y\rangle) u_{t^{-1}}\end{aligned}$$

for all $x, y \in X$ and for all $t \in G$, and from

$$\varphi(\langle X, X\rangle) B = \Phi(X)^* \Phi(X) B$$

we deduce that if Φ is nondegenerate, then φ is nondegenerate.

Lemma 3.4. *Let Φ be a nondegenerate morphism from X to $M(Y)$ and let u be a strict continuous group morphism from G to $\mathcal{U}(M(B))$ such that (φ, u) is a covariant morphism from A to $M(B)$. Then there is a unique strict continuous group morphism v from G to $\mathcal{U}M(K(Y))$ such that (v, Φ, u) is a nondegenerate covariant morphism from X to $M(Y)$.*

Proof. Let $t \in G$. From

$$\begin{aligned}\langle \Phi(\eta_t(x))\, u_t b,\, \Phi(\eta_t(y))\, u_t c\rangle &= \left\langle u_t b,\, \Phi(\eta_t(x))^* \,\Phi(\eta_t(y))\, u_t c\right\rangle \\ &= \left\langle u_t b, \varphi\left(\alpha_t^\eta(\langle x, y\rangle)\right) u_t c\right\rangle \\ &= \langle u_t b, u_t \varphi(\langle x, y\rangle)\, c\rangle \\ &= \langle b, \langle \Phi(x), \Phi(y)\rangle\, c\rangle = \langle \Phi(x)b, \Phi(y)c\rangle\end{aligned}$$

for all $x, y \in X$ and for all $b, c \in B$, and taking into account that $[\Phi(X)B] = Y$, we deduce that there is a unitary operator v_t from Y to Y such that

$$v_t(\Phi(x)b) = \Phi(\eta_t(x))\, u_t b.$$

It is clear that $v_e = \mathrm{id}_Y$, and $v_t v_s = v_{ts}$ for all $t, s \in G$. Therefore, $t \mapsto v_t$ is a group morphism from G to $\mathcal{U}M(K(Y))$. From

$$\begin{aligned}\|v_t(y) - y\| \le{}& \|v_t(y - \Phi(x)b)\| + \|\Phi(\eta_t(x) - x)\, u_t b\| \\ &+ \|\Phi(x)(u_t b - b)\| + \|y - \Phi(x)b\| \\ \le{}& \|y - \Phi(x)b\| + \|\eta_t(x) - x\|\,\|b\| + \|x\|\,\|u_t b - b\| + \|y - \Phi(x)\, b\|\end{aligned}$$

for all $t \in G$, for all $y \in Y$, for all $x \in X$ and for all $b \in B$, and taking into account that the maps $t \mapsto \eta_t(x)$ and $t \mapsto u_t b$ are continuous and $[\Phi(X)B] = Y$, we deduce that the map $t \mapsto v_t(y)$ is continuous.

Let $t \in G,\ x \in X, b \in B$. Then

$$v_t \Phi(x) u_{t^{-1}} b = \Phi(\eta_t(x))\, u_t u_{t^{-1}} b = \Phi(\eta_t(x))\, b.$$

Therefore, (v, Φ, u) is a nondegenerate covariant morphism from X to $M(Y)$.

If w is another strict continuous group morphism from G to $\mathcal{U}M(K(Y))$ such that $w_t \Phi(x) u_{t^{-1}} = \Phi(\eta_t(x))$ for all $t \in G$ and for all $x \in X$, then

$$w_t \Phi(x)b = \Phi(\eta_t(x))\, u_t b = v_t \Phi(x)b$$

for all $t \in G$, for all $x \in X$ and for all $b \in B$, and since $[\Phi(X)B] = Y$, $w_t = v_t$ for all $t \in G$. $\square$

Let (G, η, X) be a dynamical system on Hilbert C^*-modules. The linear space $C_c(G, X)$ of all continuous functions from G to X with compact support has a structure of pre-Hilbert $G \times_{\alpha^\eta} A$-module with the action of $G \times_{\alpha^\eta} A$ on $C_c(G, X)$ given by

$$(\widehat{x} f)(s) = \int_G \widehat{x}(t)\alpha_t^\eta\left(f\left(t^{-1}s\right)\right) dt$$

for all $\widehat{x} \in C_c(G, X)$ and $f \in C_c(G, A)$ and the inner product given by

$$\langle \widehat{x}, \widehat{y}\rangle(s) = \int_G \left\langle \eta_{t^{-1}}(\widehat{x}(t)), \eta_{t^{-1}}(\widehat{y}(ts))\right\rangle dt.$$

The crossed product of X by η, denoted by $G \times_\eta X$, is the Hilbert $G \times_{\alpha^\eta} A$-module obtained by the completion of the pre-Hilbert $G \times_{\alpha^\eta} A$-module $C_c(G, X)$ [4,5].

It is well known that given a C^*-dynamical system (G, α, A), then A can be embedded into $M(G \times_\alpha A)$, this embedding being given by $i_A(a)(f)(s) = af(s)$ for all $a \in A$, for all $f \in C_c(G, A)$ and for all $s \in G$, and G is isomorphic to a group of unitaries in $M(G \times_\alpha A)$, this isomorphism being given by $i_G(t)(f)(s) = \alpha_t\left(f\left(t^{-1}s\right)\right)$ for all $f \in C_c(G, A)$ and for all $t, s \in G$ (see, for example, [9, Proposition 2.34]).

In the following theorem we extend these results in the context to Hilbert C^*-modules.

Theorem 3.5. *Let (G, η, X) be a dynamical system on Hilbert C^*-modules. Then there is a nondegenerate covariant morphism $\left(i_G^X, i_X, i_G\right)$ from X to $M\left(G \times_\eta X\right)$. Moreover, the maps i_G^X, i_X and i_G are injective.*

Proof. Let $x \in X$. Define a map $i_X(x) : C_c(G, A) \to C_c(G, X)$ by

$$i_X(x)(f)(s) = xf(s)$$

for all $s \in G$. Clearly, $i_X(x)$ is linear. Let $f \in C_c(G, A)$. From

$$\begin{aligned}\langle i_X(x)(f), i_X(x)(f)\rangle(s) &= \int_G \alpha^\eta_{t^{-1}}\left(\langle i_X(x)(f)(t), i_X(x)(f)(ts)\rangle\right) dt\\ &= \int_G \alpha^\eta_{t^{-1}}\left(\langle xf(t), xf(ts)\rangle\right) dt\\ &= \int_G \alpha^\eta_{t^{-1}}\left(f(t)\right)^* \alpha^\eta_{t^{-1}}\left(\langle x, x\rangle f(ts)\right) dt\\ &= \int_G f^\#(g)\alpha^\eta_g\left(i_A\left(\langle x, x\rangle\right)(f)\left(g^{-1}s\right)\right) dg\\ &= \left(f^\# * i_A\left(\langle x, x\rangle\right)(f)\right)(s)\end{aligned}$$

for all $s \in G$, we deduce that

$$\langle i_X(x)(f), i_X(x)(f)\rangle = f^\# * i_A\left(\langle x, x\rangle\right)(f)$$

and then

$$\begin{aligned}\|i_X(x)(f)\|^2 &= \|\langle i_X(x)(f), i_X(x)(f)\rangle\| = \left\|f^\# * i_A\left(\langle x, x\rangle\right)(f)\right\|\\ &\leq \|f\| \, \|i_A\left(\langle x, x\rangle\right)(f)\| \leq \|f\|^2 \, \|\langle x, x\rangle\| = \|f\|^2 \, \|x\|^2 .\end{aligned}$$

Therefore, $i_X(x)$ is continuous and it extends to a linear map, denoted also by $i_X(x)$, from $G \times_{\alpha^\eta} A$ to $G \times_\eta X$.

Consider the map $i_X(x)^* : C_c(G, X) \to C_c(G, A)$ defined by

$$i_X(x)^*(h)(s) = \langle x, h(s)\rangle$$

for all $s \in G$. Clearly, $i_X(x)^*$ is linear. Moreover, we have

$$\begin{aligned}\langle i_X(x)(f), h\rangle(s) &= \int_G \alpha^\eta_{t^{-1}}\left(\langle xf(t), h(ts)\rangle\right) dt\\ &= \int_G \alpha^\eta_{t^{-1}}\left(f(t)^*\right) \alpha^\eta_{t^{-1}}\left(\langle x, h(ts)\rangle\right) dt\\ &= \int_G \Delta(g)^{-1}\alpha^\eta_g\left(f\left(g^{-1}\right)^*\right) \alpha^\eta_g\left(\left\langle x, h\left(g^{-1}s\right)\right\rangle\right) dg\end{aligned}$$

$$= \int_G f^{\#}(g)\alpha_g^{\eta}\left(i_X(x)^*(h)\left(g^{-1}s\right)\right)dg$$
$$= \left(f^{\#} * (i_X(x))^*(h)\right)(s) = \left\langle f, (i_X(x))^*(h)\right\rangle(s)$$

for all $f \in C_c(G, A)$, $h \in C_c(G, X)$ and for all $s \in G$. Then

$$\begin{aligned}\left\|i_X(x)^*(h)\right\|^2 &= \left\|\left\langle h, i_X(x)\left(i_X(x)^*(h)\right)\right\rangle\right\| \\ &\leq \|h\|\left\|i_X(x)\left(i_X(x)^*(h)\right)\right\| \leq \|h\|\,\|x\|\left\|i_X(x)^*(h)\right\|\end{aligned}$$

for all $h \in C_c(G, X)$, and so $i_X(x)^*$ extends by continuity to a linear map, denoted also by $i_X(x)^*$, from $G \times_\eta X$ to $G \times_{\alpha^\eta} A$. Moreover, since

$$\langle i_X(x)(f), h\rangle = \left\langle f, (i_X(x))^*(h)\right\rangle$$

for all $f \in C_c(G, A)$ and $h \in C_c(G, X)$, $i_X(x) \in M\left(G \times_\eta X\right)$. Therefore, the map $i_X(x)$ is well defined.

Let $x, y \in X$. From

$$\begin{aligned}\langle i_X(x), i_X(y)\rangle(f)(s) &= \left(i_X(x)^* \circ i_X(y)\right)(f)(s) = \langle x, i_X(y)(f)(s)\rangle \\ &= \langle x, yf(s)\rangle = \langle x, y\rangle f(s) = i_A(\langle x, y\rangle)(f)(s)\end{aligned}$$

for all $f \in C_c(G, A)$ and for all $s \in G$, we deduce that $\langle i_X(x), i_X(y)\rangle = i_A(\langle x, y\rangle)$. Thus, we showed that i_X is a morphism of Hilbert C^*-modules, and since the C^*-morphism i_A associated to i_X is injective, i_X is injective. From

$$i_X(x)(a \otimes f) = xa \otimes f$$

and

$$i_X(x)^*(y \otimes f) = \langle x, y\rangle \otimes f$$

for all $x, y \in X$, for all $a \in A$ and for $f \in C_c(G)$, and taking into account that XA, $X \otimes_{\text{alg}} C_c(G)$ and $A \otimes_{\text{alg}} C_c(G)$ are dense in X, respectively $G \times_\eta X$, respectively $G \times_{\alpha^\eta} A$, we deduce that i_X is nondegenerate.

Since (i_A, i_G) is a covariant morphism from A to $M(G \times_{\alpha^\eta} A)$, by Lemma 3.4, there is a unique strict continuous group morphism i_G^X from G to $\mathcal{U}\left(M\left(K(G \times_\eta X)\right)\right)$ such that $\left(i_G^X, i_X, i_G\right)$ is a nondegenerate covariant morphism from X to $M\left(G \times_\eta X\right)$. Moreover, since

$$\begin{aligned}i_G^X(t)(i_X(x)(a \otimes f))(s) &= (i_X(\eta_t(x))\, i_G(t)(a \otimes f))(s) \\ &= \eta_t(x)\alpha_t^{\eta}\left(af\left(t^{-1}s\right)\right) = \eta_t\left(xaf\left(t^{-1}s\right)\right)\end{aligned}$$

for all $x \in X$, for all $a \in A$ and for all $f \in C_c(G)$, we have

$$i_G^X(t)(x \otimes f)(s) = \eta_t(x) f\left(t^{-1}s\right)$$

for all $x \in X$ and for all $f \in C_c(G)$.

To show that i_G^X is injective, let $t_0 \in G$ such that $i_G^X(t_0) = \mathrm{id}_{G \times_\eta X}$. Then $\eta_{t_0}(x)\, f(e) = xf(t_0)$ for all $x \in X$ and for all $f \in C_c(G)$. Suppose that $t_0 \neq e$. Then, there is $f \in C_c(G)$ such that $f(e) = 1$ and $f(t_0) = 0$, and so $\eta_{t_0}(x) = 0$ for all $x \in X$, a contradiction, since η_{t_0} is an isomorphism of C^*-modules. Therefore, $t_0 = e$ and so i_G^X is injective. □

Remark 3.6. Suppose that A is a unital C^*-algebra and G is a discrete group. Then $G \times_{\alpha^\eta} A$ is a unital C^*-algebra and thus $M(G \times_\eta X)$ can be identified to $G \times_\eta X$. So, if A is unital and G is a discrete group, then X can be identified to a Hilbert C^*-submodule of $G \times_\eta X$ and G with a group of unitaries in $L\left(G \times_\eta X\right)$.

A covariant representations of X on the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ is a covariant morphism from X to $M(K(\mathcal{H}, \mathcal{K})) = L(\mathcal{H}, \mathcal{K})$.

Clearly, any covariant representation $(\pi, u, \mathcal{H})$ of the C^*-dynamical system (G, α, A) can be regarded as a covariant representation $(v, \pi_X, u, \mathcal{H}, \mathcal{K})$, where $\pi_X = \pi, \mathcal{K} = \mathcal{H}$ and $v = u$, of the dynamical system on Hilbert C^*-modules (G, α, A).

Example 3.7. Let π_X be a representation of X on the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. Then the map $\widetilde{\pi_X} : X \to L\left(L^2(G, \mathcal{H}), L^2(G, \mathcal{K})\right)$ defined by

$$\widetilde{\pi_X}(x)(\xi)(t) = \pi_X\left(\eta_{t^{-1}}(x)\right)(\xi(t))$$

is a representation of X, since

$$\begin{aligned}
\langle \widetilde{\pi_X}(x), \widetilde{\pi_X}(y)\rangle(\xi)(t) &= \left(\widetilde{\pi_X}(x)^* \widetilde{\pi_X}(y)\right)(\xi)(t) \\
&= \pi_X\left(\eta_{t^{-1}}(x)\right)^* \left(\widetilde{\pi_X}(y)(\xi)(t)\right) \\
&= \pi_X\left(\eta_{t^{-1}}(x)\right)^* \left(\pi_X\left(\eta_{t^{-1}}(y)\right)(\xi(t))\right) \\
&= \pi_A\left(\left\langle \eta_{t^{-1}}(x), \eta_{t^{-1}}(y)\right\rangle\right)(\xi(t)) \\
&= \pi_A\left(\alpha^\eta_{t^{-1}}(\langle x, y\rangle)\right)(\xi(t)) = \widetilde{\pi_A}(\langle x, y\rangle)(\xi)(t)
\end{aligned}$$

for all $\xi \in L^2(G, \mathcal{H})$ and for all $t \in G$, where $\widetilde{\pi_A}$ is the representation of A on $L^2(G, \mathcal{H})$ given by $\widetilde{\pi_A}(a)(\xi)(t) = \pi_A\left(\alpha^\eta_{t^{-1}}(a)\right)(\xi(t))$. Moreover, if π_X is nondegenerate, then $\widetilde{\pi_X}$ is nondegenerate, since

$$\widetilde{\pi_X}(X)\left(C_c(G) \otimes_{\mathrm{alg}} \mathcal{H}\right) = C_c(G) \otimes_{\mathrm{alg}} \pi_X(X)\mathcal{H}$$

and

$$\widetilde{\pi_X}(X)^*\left(C_c(G) \otimes_{\mathrm{alg}} \mathcal{K}\right) = C_c(G) \otimes_{\mathrm{alg}} \pi_X(X)^* \mathcal{K}$$

and since the algebraic tensor product $C_c(G) \otimes_{\mathrm{alg}} \mathcal{H}$ *is dense in* $L^2(G, \mathcal{H})$ for all Hilbert space $\mathcal{H}$.

Let $t \mapsto (\lambda_{\mathcal{H}})_t$ and $t \mapsto (\lambda_K)_t$ be the unitary representations of G on $L^2(G, \mathcal{H})$ and $L^2(G, \mathcal{K})$, given by $(\lambda_{\mathcal{H}})_t(\xi)(s) = \xi\left(t^{-1}s\right)$ for all $\xi \in L^2(G, \mathcal{H})$ and $s \in G$, respectively $(\lambda_{\mathcal{K}})_t(\zeta)(s) = \zeta\left(t^{-1}s\right)$ for all $\zeta \in L^2(G, \mathcal{K})$ and $s \in G$. Then

$$\begin{aligned}
(\lambda_{\mathcal{K}})_t \widetilde{\pi_X}(x)(\lambda_{\mathcal{H}})_{t^{-1}}(\xi)(s) &= \widetilde{\pi_X}(x)(\lambda_{\mathcal{H}})_{t^{-1}}(\xi)\left(t^{-1}s\right) \\
&= \pi_X\left(\eta_{s^{-1}t}(x)\right)\left((\lambda_{\mathcal{H}})_{t^{-1}}(\xi)\left(t^{-1}s\right)\right) \\
&= \pi_X\left(\eta_{s^{-1}t}(x)\right)(\xi(s)) \\
&= \widetilde{\pi_X}(\eta_t(x))(\xi)(s)
\end{aligned}$$

for all $\xi \in L^2(G, \mathcal{H})$ and for all $s \in G$. Therefore, $\left(\lambda_{\mathcal{K}}, \widetilde{\pi_X}, \lambda_{\mathcal{H}}, L^2(G, \mathcal{H}), L^2(G, \mathcal{K})\right)$ is a covariant representation of (G, η, X).

Two covariant representations $(v, \pi_X, u, \mathcal{H}, \mathcal{K})$ and $\left(v', \pi'_X, u', \mathcal{H}', \mathcal{K}'\right)$ are *unitarily equivalent* if there are two unitary operators $U_1 \in L(\mathcal{H}, \mathcal{H}^{'})$ and $U_2 \in L(\mathcal{K}, \mathcal{K}^{'})$ such that $\pi'_X(x)U_1 = U_2\pi_X(x)$ for all x in X, $U_1u_t = u'_tU_1$ and $U_2v_t = v'_tU_2$ for all $t \in G$.

A pair of Hilbert subspaces $(\mathcal{H}_0, \mathcal{K}_0), \mathcal{H}_0 \subseteq \mathcal{H}$ and $\mathcal{K}_0 \subseteq \mathcal{K}$, is invariant under $(v, \pi_X, u, \mathcal{H}, \mathcal{K})$ if $\pi_X(X)\mathcal{H}_0 \subseteq \mathcal{K}_0, \pi_X(X)^*\mathcal{K}_0 \subseteq \mathcal{H}_0, u_t\mathcal{H}_0 \subseteq \mathcal{H}_0$ and $v_t\mathcal{K}_0 \subseteq \mathcal{K}_0$ for all $t \in G$. A covariant representation $(v, \pi_X, u, \mathcal{H}, \mathcal{K})$ is *irreducible* if $(\mathcal{H}, \mathcal{K})$ and $(\{0\}, \{0\})$ the only invariant pairs.

In the following proposition we show that there is a bijective correspondence between nondegenerate representations of (G, η, X) and nondegenerate representations of $G \times_\eta X$.

Proposition 3.8. *Let (G, η, X) be a dynamical system on Hilbert C^*-modules and let $(v, \pi_X, u, \mathcal{H}, \mathcal{K})$ be a (nondegenerate) covariant representation of (G, η, X). Then, the map $\pi_X \times u : G \times_\eta X \to L(\mathcal{H}, \mathcal{K})$ defined by*

$$(\pi_X \times u)\left(\widehat{x}\right) = \int_G \pi_X\left(\widehat{x}(t)\right) u_t dt$$

$\widehat{x} \in C_c(G, X)$ is a (nondegenerate) representation of $G \times_\eta X$ on the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$.

The map $(v, \pi_X, u, \mathcal{H}, \mathcal{K}) \mapsto (\pi_X \times u, \mathcal{H}, \mathcal{K})$ is a bijective correspondence between nondegenerate covariant representations of (G, η, X) and nondegenerate representations of $G \times_\eta X$ which preserves the irreducibility and unitary equivalence.

Proof. By [5, Lemma 2.8], $(\pi_X \times u, \mathcal{H}, \mathcal{K})$ is a representation of $G \times_\eta X$ with the underlying $*$-representation $(\pi_A \times u, \mathcal{H})$. If $(v, \pi_X, u, \mathcal{H}, \mathcal{K})$ is nondegenerate, then $(\pi_A, u, \mathcal{H})$ is nondegenerate and so $(\pi_A \times u, \mathcal{H})$ is nondegenerate. Let $f \in C_c(G, A)$ and $x \in X$. Then $f_x \in C_c(G, X)$, where $f_x(s) = xf(s)$, and

$$\begin{aligned}(\pi_X \times u)(f_x) &= \int_G \pi_X(xf(t))\, u_t dt = \int_G \pi_X(x)\, \pi_A(f(t))\, u_t dt \\ &= \pi_X(x)\,(\pi_A \times u)(f)\end{aligned}$$

and

$$(\pi_X \times u)(f_x)^* = (\pi_A \times u)(f)^* \pi_X(x)^*.$$

Therefore, $\left[(\pi_X \times u)\left(G \times_\eta X\right)\mathcal{H}\right] = \mathcal{K}$ and $\left[(\pi_X \times v)\left(G \times_\eta X\right)^* \mathcal{K}\right] = \mathcal{H}$, and so $(\pi_X \times u, \mathcal{H}, \mathcal{K})$ is nondegenerate. Moreover, if $(v, \pi_X, u, \mathcal{H}, \mathcal{K})$ is nondegenerate, then from

$$\begin{aligned}\overline{\pi_X \times u}\,(i_X(x))\,(\pi_A \times u)(f) = \pi_X \times u\,(i_X(x)(f)) &= \int_G \pi_X(xf(t))\, u_t dt \\ &= \pi_X(x) \int_G \pi_A(f(t))\, u_t dt = \pi_X(x)\,(\pi_A \times u)(f)\end{aligned}$$

for all $f \in C_c(G, A)$ and for all $x \in X$, we deduce that $\overline{\pi_X \times u} \circ i_X = \pi_X$.

Let $\left(\pi_{G \times_\eta X}, \mathcal{H}, \mathcal{K}\right)$ be a nondegenerate representation of $G \times_\eta X$. By Lemma 2.1, it extends to a unique representation $\left(\overline{\pi_{G \times_\eta X}}, \mathcal{H}, \mathcal{K}\right)$ of $M(G \times_\eta X)$, and by [9, Proposition 2.40] there is a unique nondegenerate covariant representation $(\pi_A, u, \mathcal{H})$ of (G, α^η, A) such that $\pi_A \times u = \pi_{G \times_{\alpha^\eta} A}$. Moreover, $\pi_A = \overline{\pi_{G \times_{\alpha^\eta} A}} \circ i_A$ and $u = \overline{\pi_{G \times_{\alpha^\eta} A}} \circ i_G$.

Let $\pi_X = \overline{\pi_{G\times_\eta X}} \circ i_X$. Then $(\pi_X, \mathcal{H}, \mathcal{K})$ is a nondegenerate representation of X, and moreover, $(\pi_A, u, \mathcal{H})$ is a covariant representation of (G, α^η, A) such that

$$\langle \pi_X(x), \pi_X(y)\rangle = \pi_A(\langle x, y\rangle)$$

for all $x, y \in X$. From this fact and Lemma 3.4, we deduce that there is a unique unitary representation v of G on $G\times_\eta X$ such that $(v, \pi_X, u, \mathcal{H}, \mathcal{K})$ is a nondegenerate covariant representation of (G, η, X). Moreover, since

$$\begin{aligned}
(\pi_X \times u)(x\otimes f) &= \int_G \pi_X(xf(t))\, u_t dt \\
&= \int_G \overline{\pi_{G\times_\eta X}}(i_X(xf(t)))\, \overline{\pi_{G\times_{\alpha^\eta} A}}(i_G(t))\, dt \\
&= \int_G \overline{\pi_{G\times_\eta X}}(i_X(x)\, f(t) i_G(t))\, dt \\
&= \overline{\pi_{G\times_\eta X}}\left(i_X(x)\int_G f(t) i_G(t) dt\right) \quad \text{(cf. [8, Proposition 2.31])} \\
&= \overline{\pi_{G\times_\eta X}}\left(i_X(y) i_A(\langle y, y\rangle)\int_G f(t)\, i_G(t) dt\right) \\
&\quad \text{(cf. [9, Corollary 2.36])} \\
&= \overline{\pi_{G\times_\eta X}}(i_X(y)(\langle y, y\rangle \otimes f)) = \overline{\pi_{G\times_\eta X}}(y\langle y, y\rangle \otimes f) \\
&= \pi_{G\times_\eta X}(x\otimes f)
\end{aligned}$$

for all $x \in X$ and for all $f \in C_c(G)$, and since $X \otimes_{\mathrm{alg}} C_c(G)$ is dense in $G\times_\eta X$, $\pi_X \times u = \pi_{G\times_\eta X}$.

If $(v', \pi'_X, u', \mathcal{H}, \mathcal{K})$ is another nondegenerate covariant representation of (G, η, X) such that $\pi'_X \times u' = \pi_{G\times_\eta X}$, then

$$\pi_X = \overline{\pi_{G\times_\eta X}} \circ i_X = \overline{\pi'_X \times u'} \circ i_X = \pi'_X$$

and

$$u = \overline{\pi_{G\times_{\alpha^\eta} A}} \circ i_A = \overline{\pi'_A \times u'} \circ i_A = u'.$$

Therefore, the map $(v, \pi_X, u, \mathcal{H}, \mathcal{K}) \mapsto (\pi_X \times u, \mathcal{H}, \mathcal{K})$ is a bijective correspondence between nondegenerate covariant representations of (G, η, X) and nondegenerate representations of $G\times_\eta X$.

Now, suppose that $(v, \pi_X, u, \mathcal{H}, \mathcal{K})$ is an irreducible covariant nondegenerate representation of (G, η, X) and $(\mathcal{H}_0, \mathcal{K}_0)$ is an invariant pair under $(\pi_X \times u, \mathcal{H}, \mathcal{K})$. Then, $\mathcal{H}_0$ is invariant under $(\pi_A \times u, \mathcal{H})$ and by [9, Proposition 2.40], it is invariant under π_A and u. Since

$$\pi_X(X)^*(\pi_X(X)\mathcal{H}_0) = \pi_A(\langle X, X\rangle)\,\mathcal{H}_0 \subseteq \mathcal{H}_0$$

and

$$v_t(\pi_X(X)\mathcal{H}_0) = \pi_X(\eta_t(X))\, u_t\mathcal{H}_0 \subseteq \pi_X(X)\mathcal{H}_0$$

for all $t \in G$, $(\mathcal{H}_0, [\pi_X(X)\mathcal{H}_0])$ is invariant under $(v, \pi_X, u, \mathcal{H}, \mathcal{K})$, and since $(v, \pi_X, u, \mathcal{H}, \mathcal{K})$ is irreducible, $(\mathcal{H}_0, [\pi_X(X)\mathcal{H}_0]) = (\mathcal{H}, \mathcal{K})$ or $(\mathcal{H}_0, [\pi_X(X)\mathcal{H}_0]) = (\{0\}, \{0\})$. If $(\mathcal{H}_0, [\pi_X(X)\mathcal{H}_0]) = (\mathcal{H}, \mathcal{K})$, then $\mathcal{H}_0 = \mathcal{H}$ and $\mathcal{K}_0 = \mathcal{K}$, since

$$\mathcal{K} = \left[(\pi_X \times u)\left(G \times_\eta X\right)\mathcal{H}_0\right] \subseteq \mathcal{K}_0.$$

If $(\mathcal{H}_0, [\pi_X(X)\mathcal{H}_0]) = (\{0\}, \{0\})$, then $\mathcal{H}_0 = \{0\}$ and so $(\pi_X \times u)\left(G \times_\eta X\right)^* \mathcal{K}_0 = \{0\}$. From

$$\langle k_0, (\pi_X \times u)(z)h\rangle = \left\langle(\pi_X \times u)(z)^* k_0, h\right\rangle = 0$$

for all $k_0 \in \mathcal{K}_0$, for all $h \in \mathcal{H}$ and for all $z \in G \times_\eta X$, and taking into account that $[\pi_X(X)\mathcal{H}] = \mathcal{K}$, we conclude that $\mathcal{K}_0 = \{0\}$. Therefore, $(\pi_X \times u, \mathcal{H}, \mathcal{K})$ is irreducible. Conversely, suppose that $(\pi_X \times u, \mathcal{H}, \mathcal{K})$ is irreducible, and $(\mathcal{H}_0, \mathcal{K}_0)$ is invariant under $(v, \pi_X, u, \mathcal{H}, \mathcal{K})$. Then $(\mathcal{H}_0, \mathcal{K}_0)$ is invariant under $(\pi_X \times u, \mathcal{H}, \mathcal{K})$, since

$$\langle(\pi_X \times u)(f)h_0, k\rangle = \int_G \langle \pi_X(f(t))\, v_t h_0, k\rangle\, dt = 0$$

for all $h_0 \in \mathcal{H}_0$ and for all $k \in \mathcal{K}_0^{\perp}$, and since

$$\begin{aligned}\left\langle(\pi_X \times u)(f)^* k_0, h\right\rangle &= \int_G \langle k_0, \pi_X(f(t))\, u_t h\rangle\, dt \\ &= \int_G \left\langle u_{t^{-1}} \pi_X(f(t))^* k_0, h\right\rangle dt = 0\end{aligned}$$

for all $k_0 \in \mathcal{K}_0$ and for all $h \in \mathcal{H}_0^{\perp}$. Therefore $(v, \pi_X, u, \mathcal{H}, \mathcal{K})$ is irreducible.

Suppose that $(v, \pi_X, u, \mathcal{H}, \mathcal{K})$ and $\left(v', \pi'_X, u', \mathcal{H}', \mathcal{K}'\right)$ are two unitarily equivalent nondegenerate covariant representations of (G, η, X). If U_1 and U_2 are two unitary operators in $L(\mathcal{H}, \mathcal{H}')$, respectively $L(\mathcal{K}, \mathcal{K}')$ such that $U_2\pi_X(x) = \pi'_X(x)U_1$ for all x in X, $U_1 u_t = u'_t U_1$ and $U_2 v_t = v'_t U_2$ for all $t \in G$, then

$$\begin{aligned}U_2(\pi_X \times u)(f) &= U_2 \int_G \pi_X(f(t))\, u_t dt = \int_G U_2 \pi_X(f(t))\, u_t dt \\ &= \int_G \pi'_X(f(t))\, u'_t U_1 dt = \left(\pi'_X \times u'\right)(f)U_1\end{aligned}$$

for all $f \in C_c(G, X)$ and so $(\pi_X \times u, \mathcal{H}, \mathcal{K})$ and $\left(\pi'_X \times u', \mathcal{H}', \mathcal{K}'\right)$ are unitarily equivalent. Conversely, suppose that $(\pi_X \times u, \mathcal{H}, \mathcal{K})$ and $\left(\pi'_X \times u', \mathcal{H}', \mathcal{K}'\right)$ are unitarily equivalent. Let U_1 and U_2 be two unitary operators in $L(\mathcal{H}, \mathcal{H}')$, respectively $L(\mathcal{K}, \mathcal{K}')$ such that $U_2(\pi_X \times u)(z) = \left(\pi'_X \times u'\right)(z)U_1$ for all $z \in G \times_\eta X$. Then $U_2\overline{\pi_X \times u}(z') = \overline{\pi'_X \times u'}(z')U_1$ for all $z' \in M(G \times_\eta X)$ and so $U_2\pi_X(x) = \pi'_X(x)U_1$ for all x in X and $U_1 u_t = u'_t U_1$ for all $t \in G$. Moreover, since

$$\begin{aligned}U_2 v_t \pi_X(x) &= U_2\overline{\pi_X \times u}\left(i_G^X(t) i_X(x)\right) = \overline{\pi'_X \times u'}\left(i_G^X(t) i_X(x)\right) U_1 \\ &= v'_t \pi'_X(x) U_1 = v'_t U_2 \pi_X(x)\end{aligned}$$

and since $[\pi_X(X)\mathcal{H}] = \mathcal{K}$, we have $U_2 v_t = v'_t U_2$ for all $t \in G$. Therefore, $(v, \pi_X, u, \mathcal{H}, \mathcal{K})$ and $\left(v', \pi'_X, u', \mathcal{H}', \mathcal{K}'\right)$ are unitarily equivalent, and the proposition is proved. □

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